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On the Zeros of $\zeta(s)$ and $\zeta'(s)$

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In this paper, we will show that there is a close connection between the vertical distribution of the zeros of the Riemann Zeta function $\zeta(s)$ and that of the zeros of its derivative $\zeta'(s)$. To be precise, we will prove, assuming the Riemann Hypothesis, the following theorem:

THEOREM. *Let T_0 be a large fixed positive real number and $T \geqslant T_0$. Let $\rho'_0 = \beta'_0 + i\gamma'_0$ be a zero of $\zeta'(s)$ with $1/2 < \beta'_0 < 1/2 + g(T)$ where $g(T) \to 0$ when $T \to \infty$, and $T \leqslant \gamma'_0 \leqslant 2T$. Suppose there exists another zero ρ'_1 of $\zeta'(s)$ with $|\rho'_1 - \rho'_0| \leqslant A(\beta'_0 - 1/2)$ for some absolute constant $A > 0$. Then there exists a positive real number B depending only on A and a zero $\rho = \beta + i\gamma$ of $\zeta(s)$ with $|\gamma'_0 - \gamma| \leqslant B(\beta'_0 - 1/2)$. (Note that here ρ'_1 may be equal to ρ'_0, i.e., $\zeta'(s)$ may have a double zero at $s = \rho'_0$.)*

We will also give a slight generalization of this result.

The study of the horizontal distribution of the zeros of $\zeta'(s)$, $\zeta'(s)$ being the derivative of the Riemann Zeta function $\zeta(s)$, is an important area in Number Theory. It is now known that this problem is closely related to that concerning the zeros of $\zeta(s)$ (see, for example, [4]). In fact, important results concerning $\zeta(s)$ have already been obtained by exploiting such relationship (see, for example, [3]). The objective of this paper is to show that there is also a close connection between the vertical distribution of zeros of $\zeta'(s)$ and that of $\zeta(s)$. We shall prove, assuming the Riemann Hypothesis (RH), that

THEOREM 1. *(Assume RH) Let T_0 be a large fixed positive real number and $T \geqslant T_0$. Let $\rho'_0 = \beta'_0 + i\gamma'_0$ be a zero of $\zeta'(s)$ with $1/2 < \beta'_0 < 1/2 + g(T)$ where $g(T) \to 0$ when $T \to \infty$, and $T \leqslant \gamma'_0 \leqslant 2T$. Suppose there exists another zero ρ'_1 of $\zeta'(s)$ with $|\rho'_1 - \rho'_0| \leqslant A(\beta'_0 - 1/2)$ for some absolute constant $A > 0$. Then there exists a positive real number B depending only on A and a zero $\rho = \beta + i\gamma$ of $\zeta(s)$ with $|\gamma'_0 - \gamma| \leqslant B(\beta'_0 - 1/2)$. (Note that here ρ'_1 may be equal to ρ'_0, i.e., $\zeta'(s)$ may have a double zero at $s = \rho'_0$.)*

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Remarks. We remark here that, following our proof of Theorem 1, one can see that B can be any real number satisfying $B > 4(A+1)$. A more careful analysis may yield a better range of values of B. On the other hand, as one can see from the proof, it is still necessary that $B > 1$.

This theorem seems to be the first of its kind known to date. It illustrates the possibility of obtaining results concerning the vertical distribution of zeros of $\zeta(s)$ from information concerning that of $\zeta'(s)$. Unfortunately, this latter area of research seems to be relative new and it seems that no non-trivial result has yet been proven.

Following our proof of Theorem 1, we can also show that:

THEOREM 2. *(Assume RH) Let $T \geqslant T_0$, T_0 an absolute large constant. Suppose $t \in [T, 2T]$ and $\sigma \in (1/2, 1/2 + g(T)]$, $g(T) \to 0$ as $T \to \infty$, are such that*

$$\left|\frac{\zeta'}{\zeta}(\sigma + it)\right| < \frac{1}{4}\log t$$

and

$$\left|\frac{d}{ds}\frac{\zeta'}{\zeta}(\sigma + it)\right| < \frac{1}{8}\frac{\log t}{\sigma - 1/2}.$$

Then there exists a constant A such that there exists a non-trivial zero ρ of $\zeta(s)$ with $|\gamma - t| \leqslant A(\sigma - 1/2)$.

This theorem may be more useful in some applications.

Proof of Theorem 1. Our proof makes use of the well known formula

$$\frac{\zeta'(s)}{\zeta(s)} = B - \frac{1}{s-1} + \frac{1}{2}\log\pi - \frac{1}{2}\frac{\Gamma'(s/2+1)}{\Gamma(s/2+1)} + \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right). \tag{1}$$

(See Chapter 12 of [2]). Now suppose $\rho'_0 = \beta'_0 + i\gamma'_0$ and $\rho'_1 = \beta'_1 + i\gamma'_1$ are two zeros of $\zeta'(s)$ with $|\rho'_1 - \rho'_0| \leqslant A(\beta'_0 - 1/2)$ for some $A > 0$. We shall also suppose $\rho'_1 \neq \rho'_0$. We then have

$$\begin{aligned} 0 &= \frac{\zeta'(\rho'_0)}{\zeta(\rho'_0)} - \frac{\zeta'(\rho'_1)}{\zeta(\rho'_1)} \\ &= \left(\frac{1}{\rho'_1 - 1} - \frac{1}{\rho'_0 - 1}\right) + \frac{1}{2}\left(\frac{\Gamma'(\rho'_1/2+1)}{\Gamma(\rho'_1/2+1)} - \frac{\Gamma'(\rho'_0/2+1)}{\Gamma(\rho'_0/2+1)}\right) \\ &\quad + \sum_{\rho}\left(\frac{1}{\rho'_0 - \rho} - \frac{1}{\rho'_1 - \rho}\right). \end{aligned} \tag{2}$$

The first two terms on RHS of (2) are $\ll |\rho'_0 - \rho'_1|/T^2$. To estimate the third and fourth terms, we use the following version of Stirling formula which holds for $\Re w > 0$:

$$\frac{\Gamma'(w)}{\Gamma(w)} = C + \log w - \frac{1}{2w} - \int_0^\infty \frac{2\eta}{\eta^2 + w^2} \frac{d\eta}{e^{2\pi\eta} - 1}$$

where C is an absolute constant (see Chapter 5 of [1]). Since $\beta'_0 - 1/2 \leqslant g(T)$ where $g(T) \to 0$ when $T \to \infty$, as long as T is sufficiently large, both $\Re(\rho'_1/2 + 1)$ and $\Re(\rho'_0/2 + 1)$ are >0. This formula is therefore applicable and we can see that these two terms of (2) are $\ll |\rho'_0 - \rho'_1|/T$. We therefore see that

$$\sum_\rho \left(\frac{1}{\rho'_1 - \rho} - \frac{1}{\rho'_0 - \rho} \right) \ll \frac{|\rho'_0 - \rho'_1|}{T}.$$

Or equivalently,

$$\sum_\rho \frac{1}{(\rho'_1 - \rho)(\rho'_0 - \rho)} \ll \frac{1}{T}. \tag{3}$$

We now assume that there is no zero $\rho = \beta + i\gamma$ of $\zeta(s)$ with $|\gamma - \gamma'_0| \leqslant B(\beta'_0 - 1/2)$ where B is large compared to A (its value will be clear from our proof). The real part of LHS of (3) is equal to

$$\sum_\rho \frac{(\beta'_1 - 1/2)(\beta'_0 - 1/2) - (\gamma'_0 - \gamma)(\gamma'_1 - \gamma)}{\{(\beta'_0 - 1/2)^2 + (\gamma'_0 - \gamma)^2\}\{(\beta'_1 - 1/2)^2 + (\gamma'_1 - \gamma)^2\}}.$$

Note that since there is no zero of $\zeta(s)$ with $|\gamma - \gamma'_0| \leqslant B(\beta'_0 - 1/2)$ and since $|\rho'_0 - \rho'_1| \leqslant A(\beta' - 1/2)$, as long as $B \geqslant 2A$, all zeros of $\zeta(s)$ either have $\gamma \geqslant \max(\gamma'_1, \gamma'_0)$ or $\gamma \leqslant \min(\gamma'_1, \gamma'_0)$. So, the expression $-(\gamma'_0 - \gamma)(\gamma'_1 - \gamma)$ is always negative. Moreover, with our assumption on the zeros of $\zeta(s)$, $|\gamma'_0 - \gamma|\,|\gamma'_1 - \gamma| \geqslant B^2(\beta'_0 - 1/2)^2/2$. Now since

$$\begin{aligned} 0 &< (\beta'_1 - 1/2)(\beta'_0 - 1/2) = (\beta'_1 - \beta'_0 + \beta'_0 - 1/2)(\beta'_0 - 1/2) \\ &\leqslant (A+1)(\beta'_0 - 1/2)^2 \\ &\leqslant \frac{2(A+1)}{B^2} |\gamma'_0 - \gamma|\,|\gamma'_1 - \gamma|, \end{aligned}$$

the numerator of the above sum is

$$\leqslant -\left(1 - \frac{2(A+1)}{B^2}\right)(\gamma'_0 - \gamma)(\gamma'_1 - \gamma).$$

This is <0 if B is large compared to A. So, the LHS of (3) has absolute value

$$\geqslant\left(1-\frac{2(A+1)}{B^2}\right)\sum_{\rho}\frac{(\gamma_0'-\gamma)(\gamma_1'-\gamma)}{\{(\beta_0'-1/2)^2+(\gamma_0'-\gamma)^2\}\{(\beta_1'-1/2)^2+(\gamma_1'-\gamma)^2\}}.$$

Following the same argument, the denominator of this last sum is $\leqslant 4(\gamma_0'-\gamma)^2(\gamma_1'-\gamma)^2$ (if B is large compared to A). So, the expression above is

$$\begin{aligned}\geqslant\frac{1}{8}\sum_{\rho}\frac{(\gamma_0'-\gamma)(\gamma_1'-\gamma)}{(\gamma_0'-\gamma)^2(\gamma_1'-\gamma)^2}&=\frac{1}{8}\sum_{\rho}\frac{1}{(\gamma_0'-\gamma)(\gamma_1'-\gamma)}\\ &\geqslant\frac{1}{16}\sum_{\rho}\frac{1}{(\gamma_0'-\gamma)^2}\\ &=\frac{1}{16(\beta_0'-1/2)}\sum_{\rho}\frac{\beta_0'-1/2}{(\gamma_0'-\gamma)^2}.\end{aligned}\tag{4}$$

We now assert that the last sum is $\geqslant(\log T)/4$. This will contradict (3) and complete our proof for the case $\rho_0'\neq\rho_1'$. To see this, we put $s=\rho_0'$ into (1) and take the real part on both sides of the equation. We then have

$$0=-\frac{1}{2}\Re\frac{\Gamma'(\rho_0'/2+1)}{\Gamma(\rho_0'/2+1)}+\sum_{\rho}\frac{\beta_0'-1/2}{(\beta_0'-1/2)^2+(\gamma_0'-\gamma)^2}+\mathcal{O}(1).$$

Or equivalently,

$$\sum_{\rho}\frac{\beta_0'-1/2}{(\beta_0'-1/2)^2+(\gamma_0'-\gamma)^2}=\frac{1}{2}\Re\frac{\Gamma'(\rho_0'/2+1)}{\Gamma(\rho_0'/2+1)}+\mathcal{O}(1).\tag{5}$$

The LHS of this last equation is

$$\leqslant\sum_{\rho}\frac{\beta_0'-1/2}{(\gamma_0'-\gamma)^2}.$$

So we just need to estimate the RHS. By Stirling formula (see Chapter 10 of [2])

$$\frac{\Gamma'(\omega)}{\Gamma(\omega)}=\log\omega+\mathcal{O}(|\omega|^{-1}).$$

So,

$$\frac{1}{2}\Re\frac{\Gamma'(\rho_0'/2+1)}{\Gamma(\rho_0'/2+1)}=\frac{1}{2}\log|\rho_0'/2+1|+\mathcal{O}(|\rho_0'/2+1|^{-1})\geqslant(\log T)/2+\mathcal{O}(1).$$

This completes our proof for the assertion.

The case $\rho_0'=\rho_1'$ can be handled in a similar manner. However, instead of taking the difference between $(\zeta'/\zeta)(\rho_0')$ and $(\zeta'/\zeta)(\rho_1')$, we differentiate the function $(\zeta'/\zeta)(s)$ and look at $(d/ds)(\zeta'/\zeta)(\rho_0')$. This is equal to zero since ρ_0' is a double zero. On the other hand, by using the formula in (1), one can also see that it is $\approx -\sum_\rho 1/(\rho_0'-\rho)^2$. This last expression is similar to the LHS of (3). Following analysis similar to that above, we see that the absolute value of its real part is again large (if there is no zero of $\zeta(s)$ near ρ_0'). This completes the proof of the theorem. ∎

Proof of Theorem 2. The proof of this theorem is similar to that of the case $\rho_0'=\rho_1'$ in Theorem 1. In fact one can see that we do not need the strong conditions

$$\frac{d}{ds}\frac{\zeta'}{\zeta}(\sigma+it)=\frac{\zeta'}{\zeta}(\sigma+it)=0.$$

They may be replaced by

$$\left|\frac{\zeta'}{\zeta}(\sigma+it)\right|<\frac{1}{4}\log t$$

and

$$\left|\frac{d}{ds}\frac{\zeta'}{\zeta}(\sigma+it)\right|<\frac{1}{8}\frac{\log t}{\sigma-1/2}.$$

These are precisely the assumptions in our theorem. ∎

References

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